

On the Graded Identities for Elementary Gradings in Matrix Algebras over Infinite Fields

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Abstract

We consider the algebra $E \otimes E$ over an infinite field equipped with a \mathbb{Z}_2 -grading where the canonical basis is homogeneous and prove that in various cases the graded identities are just the ordinary ones. If the grading is a non-canonical grading obtained as a quotient grading of the natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading we exhibit a basis for the graded identities.

1 Introduction

In [11] Kravovski and Regev proved that, for fields of characteristic zero, the polynomial identities of the Grassmann algebra are consequences of the identity $[x_1, x_2, x_3]$. In [8] it was proved that the last result holds for infinite fields. The identities of $E \otimes E$ were considered in [12] where A. Popov proved that the polynomials $[x_1, x_2, [x_3, x_4], x_5]$ and $[[x_1, x_2]^2, x_2]$ form a basis for $T(E \otimes E)$ if the base field is of characteristic zero. We remark that determining a basis for $E \otimes E$ over fields of positive characteristic is still an open problem.

The problem of determining a basis for the polynomial identities of a given algebra is hard and was solved in a few cases only, and it is useful to consider variations of the notion of polynomial identity such as graded polynomial identities. In [9] Kemer develops a structural theory of T-ideals used in his solution to the Specht problem in characteristic zero, and the concept of \mathbb{Z}_2 -graded identity was a key component. Graded identities have various applications in PI-Theory and shortly afterwards became an object of independent studies.

In [5] the authors proved that if G is a finite group and A is a G -graded algebra then A satisfies a polynomial identity if and only if the component A_0

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is a PI-algebra. Moreover graded identities provide a usefull way to prove that two algebras satisfy the same polynomial identities. In [17] O. M. Di Vincenzo determined a basis for the \mathbb{Z}_2 -graded identities of $M_{11}(E)$, where E denotes the Grassmann algebra of a vector space of infinite dimension over a field of characteristic zero. This results allowed him to give a new proof that over fields of characteristic zero this algebra is PI-equivalent to $E \otimes E$. This equivalence is a consequence of Kemmer's Tensor Product Theorem which was first proved by Kemmer using his structural theory of T-ideals, we mention that another proof of this theorem was given by A. Regev in [13] using graded identities and in [3] the methods developed by Regev were used to prove that a multilinear version of the Tensor Product Theorem holds when the characteristic of the field is $p > 0$.

Therefore describing the gradings and the corresponding graded identities of a given algebra is an important problem. Bases for the graded identities for the matrix algebra $M_n(K)$ were determined in [15], [16] for its natural \mathbb{Z} and \mathbb{Z}_n gradings for fields \mathbb{K} of characteristic zero. In [1], [2] it was proved that these results hold for infinite fields. Graded identities for elementary gradings of $M_n(K)$ were studied in [4] for fields of characteristic zero and in [7] for infinite fields. The \mathbb{Z}_2 -graded identities of E , for any grading where the generating vector space is graded, were described in [18] for fields of characteristic zero and in [6] for infinite fields. We mention that the graded codimensions were calculated in [14]. The graded identities of the algebra $E \otimes E$, with its canonical grading, over infinite fields were studied in [10].

In this paper we consider the \mathbb{Z}_2 -graded identities for the tensor square of the Grassmann algebra $E \otimes E$ for \mathbb{Z}_2 -gradings where each of the two E components have a grading where the generating space is homogeneous. We find a basis for the identities if the grading is a non-canonical grading obtained as a quotient grading of the natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading and prove that in various cases the graded identities are just the ordinary ones.

2 Preliminaries

In this paper \mathbb{K} denotes an infinite field of characteristic different from 2, all algebras and vector spaces will be considered over \mathbb{K} .

Let V be a vector space with countable basis $\{e_1, e_2, \dots, e_n, \dots\}$, the **Grassmann algebra** of V , wich we denote by E , is the associative algebra with basis (as a vector space) consisting of 1 and the products $e_{i_1}e_{i_2}\dots e_{i_k}$, where $i_1 < i_2 < \dots < i_k$, and multiplication determined by $e_ie_j + e_je_i = 0$.

A G -grading on an associative algebra A , where G is a group, is a decomposition $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subset A_{gh}$. We say that an element in A_g is homogeneous of degree g . If H is a normal subgroup of G the quotient G/H -grading is defined as $A = \bigoplus_{\bar{g} \in G/H} A_{\bar{g}}$, where $A_{\bar{g}} = \bigoplus_{h \in H} A_{gh}$.

In [6] the author considers gradings on E such that the vector space V is a graded subspace, these are described in terms of a mapping $\|\cdot\| : \{e_1, e_2, \dots, e_n, \dots\} \rightarrow \mathbb{Z}_2$. Given such a mapping we define

$\|e_{i_1}e_{i_2}\dots e_{i_k}\| = \|e_{i_1}\| + \dots \|e_{i_k}\|$ and this determines a grading in E . The gradings induced by the maps $\|\cdot\|_{k^*}$, $\|\cdot\|_\infty$ and $\|\cdot\|_k$ defined by

$$\|e_i\|_{k^*} = \begin{cases} 1 & \text{for } i = 1, 2, \dots, k \\ 0 & \text{otherwise,} \end{cases}$$

$$\|e_i\|_\infty = \begin{cases} 1 & \text{for } i \text{ even} \\ 0 & \text{otherwise,} \end{cases}$$

$$\|e_i\|_k = \begin{cases} 0 & \text{for } i = 1, 2, \dots, k \\ 1 & \text{otherwise,} \end{cases}$$

were considered. We denote by E_{k^*} , E_∞ and E_k respectively the Grassmann algebra with each of these gradings.

Given \mathbb{Z}_2 -graded algebras R and Q their tensor product $A = R \otimes Q$ has a natural \mathbb{Z}_2 -grading $A = A_0 \oplus A_1$, where $A_0 = (R_0 \otimes Q_0) \oplus (R_1 \otimes Q_1)$ and $A_1 = R_0 \otimes Q_1 \oplus R_1 \otimes Q_0$. We denote by $g(a)$ the degree of a homogeneous element a in this grading.

The algebra A also has a natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading, we define $A_{(i,j)} = A_i \otimes A_j$, $i, j \in \mathbb{Z}_2$. If A denotes the algebra $E \otimes E$ with its natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading and H is a proper subgroup of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ then $G/H \cong \mathbb{Z}_2$ and there are three possible quotient gradings: $E_0 \otimes E_0$ which is the canonical \mathbb{Z}_2 -grading and two non-canonical isomorphic gradings: $E_{0^*} \otimes E_0$ and $E_0 \otimes E_{0^*}$. In this paper we consider the following gradings:

- (i) $E_{0^*} \otimes E_0$ and $E_0 \otimes E_{0^*}$ - these are the non-canonical quotient gradings and are studied in Section 3 where a finite basis is determined;
- (ii) $E_\infty \otimes E_k$, $E_\infty \otimes E_{j^*}$, $E_\infty \otimes E_\infty$, $E_k \otimes E_\infty$ and $E_{j^*} \otimes E_\infty$ - these are the gradings with an E_∞ component and are studied in Section 4. We prove that the graded identities are essentially the ordinary ones,
- (iii) $E_{k^*} \otimes E_{j^*}$ - the graded identities are studied in Section 5, in this case the description of the graded identities is done modulo the ordinary ones.

Let Y and Z denote two disjoint countable sets of noncommutative variables and $\mathbb{K}\langle X \rangle$ denote the free associative algebra freely generated by X , where $X = Y \cup Z$. This algebra is 2-graded if we impose that the variables Y have degree 0 and the variables Z have degree 1. We say that a polynomial $f(x_1, \dots, x_n) \in \mathbb{K}\langle X \rangle$ is a **polynomial identity** in an associative algebra A if $f(a_1, \dots, a_n) = 0$ for any a_1, \dots, a_n in A . Moreover if $A = A_0 \oplus A_1$ is a \mathbb{Z}_2 -graded algebra a polynomial $f(y_1, \dots, y_m, z_1, \dots, z_n) \in \mathbb{K}\langle X \rangle$ is a **\mathbb{Z}_2 -graded polynomial identity** (or 2-graded polynomial identity) if $f(a_1, a_2, \dots, a_m, b_1, \dots, b_n) = 0$ for any $a_1, a_2, \dots, a_m \in A_0$ and any $b_1, \dots, b_n \in A_1$. We denote by $T(A)$ the set of all polynomial identities of the algebra A and by $T_2(A)$ the set of all 2-graded polynomial identities of the graded algebra $A = A_0 \oplus A_1$. It is well

known that the set $T(A)$ (resp. $T_2(A)$) is an ideal of $\mathbb{K}\langle X \rangle$ stable under all endomorphisms (resp. graded endomorphisms) of $\mathbb{K}\langle X \rangle$, we call such ideals T -ideals (resp. T_2 -ideals).

Let $[x_1, x_2] = x_1x_2 - x_2x_1$ be the commutator of x_1 and x_2 , we define inductively the higher commutators

$$[x_1, x_2, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n].$$

We denote by \circ the Jordan product, i. e., given u, v in an associative algebra we have $u \circ v = uv + vu$. In this paper $B(Y, Z)$ is the subalgebra of $\langle X \rangle$ generated by Z and by the non-trivial commutators. The elements of $B(Y, Z)$ are called **Y -proper polynomials**. It is well known that, since our field is infinite, all the 2-graded identities of a 2-graded algebra A follow from the Y -proper multihomogeneous ones.

2.1 The algebra $E \otimes E$

We denote by A the algebra $E \otimes E$. Let $C = \{e_i \otimes 1, 1 \otimes e_i | i = 1, 2, \dots, j = 1, 2, \dots\}$ we totally order C by imposing that $e_i \otimes 1 < 1 \otimes e_j$ for every i, j , moreover if $m < n$ then $e_m \otimes 1 < e_n \otimes 1$ and $1 \otimes e_m < 1 \otimes e_n$. Then given any $b \in \mathbf{B}$, where \mathbf{B} is the canonical basis of A , there exists a unique sequence $c_1 < c_2 < \dots < c_k$ of elements in C such that $b = c_1 \dots c_k$.

Definition 1 Let $b = c_1 \dots c_k$, where $b \in \mathbf{B}$ and $c_i \in C$. We say that the set $\{c_1, \dots, c_k\}$ is the support of b and denote it by $\text{supp}(b)$.

If $b_1, b_2 \in \mathbf{B}$ then $b_1b_2 \neq 0$ if and only if b_1 and b_2 have disjoint supports, moreover in this case $b_1b_2 \in \mathbf{B}$ or $-b_1b_2 \in \mathbf{B}$, i. e., $b_1b_2 = \pm b$, where $b \in \mathbf{B}$. Clearly the sign is determined by the degree of the elements in $\text{supp}(b_1)$ and in $\text{supp}(b_2)$ in the canonical $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of $E \otimes E$. We generalize this in the following remark.

Remark 2 Given a monomial $m(x_1, \dots, x_n) \in \mathbb{K}\langle X \rangle$ and $b_1, \dots, b_n \in \mathbf{B}$ it follows that $m(b_1, \dots, b_n) \neq 0$ if and only if the elements in b_1, \dots, b_n have disjoint supports. Moreover if $m(b_1, \dots, b_n) \neq 0$ then $m(b_1, \dots, b_n) = \pm b$, where $b \in \mathbf{B}$, and the sign is determined by the monomial and the degrees of the elements in the sets $\text{supp}(b_1), \dots, \text{supp}(b_n)$ with respect to the canonical $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of $E \otimes E$.

Proposition 3 Let $f(x_1, \dots, x_n)$ be a multihomogeneous polynomial, and let $a_l = \sum_{k=1}^{v_l} \alpha_k^l b_k^l$, $l = 1, 2, \dots, n$, where $b_k^l \in \mathbf{B}$, $\alpha_k^l \in K$, and the elements b_k^l have pairwise disjoint supports. If $c_k^l \in \mathbf{B}$ are elements in the canonical basis of A with $g(b_k^l) = g(c_k^l)$ and $a'_l = \sum_{k=1}^{n_l} \alpha_k^l c_k^l$ then $f(a_1, \dots, a_n) = 0$ implies $f(a'_1, \dots, a'_n) = 0$.

Proof. It follows from the equalities $g(b_k^l) = g(c_k^l)$ that we may write

$$f(a_1, \dots, a_n) = \sum_{i=1}^{\lambda} P_i(\alpha_1^1, \dots, \alpha_n^{v_n}) m_i(b_1^1, \dots, b_n^{v_n}),$$

and

$$f(a'_1, \dots, a'_n) = \sum_{i=1}^{\lambda} P_i(\alpha_1^1, \dots, \alpha_n^{v_n}) m_i(c_1^1, \dots, c_n^{v_n}),$$

where m_1, \dots, m_λ are monomials in $\mathbb{K}\langle X \rangle$ and P_1, \dots, P_λ are polynomials in commuting variables. Since the elements b_k^l have pairwise disjoint supports it follows from the previous remark that $\pm m_1(b_1^1, \dots, b_n^{v_n}), \dots, \pm m_\lambda(b_1^1, \dots, b_n^{v_n})$ are distinct elements of $\in \mathbf{B}$, therefore $P_i(\alpha_1^1, \dots, \alpha_n^{v_n}) = 0$, $i = 1, \dots, \lambda$, hence $f(a'_1, \dots, a'_n) = 0$. ■

3 The non-canonical quotient gradings

We denote by $A = A_{(0,0)} \oplus A_{(1,0)} \oplus A_{(0,1)} \oplus A_{(1,1)}$ the canonical $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of the algebra $E \otimes E$. There are three possible quotient gradings:

- (1) $A_0 = A_{(0,0)} \oplus A_{(1,1)}$ and $A_1 = A_{(1,0)} \oplus A_{(0,1)}$;
- (2) $A_0 = A_{(0,0)} \oplus A_{(1,0)}$ and $A_1 = A_{(0,1)} \oplus A_{(1,1)}$;
- (3) $A_0 = A_{(0,0)} \oplus A_{(0,1)}$ and $A_1 = A_{(1,0)} \oplus A_{(1,1)}$.

The grading in (1) is the usual \mathbb{Z}_2 -grading of $E \otimes E$ and the graded polynomial identities in this case were described in [10]. In this section we find a basis for the graded identities of A with the quotient grading in (2). Note that the results in this section also hold for the grading in (3).

Denote by I the ideal generated by the polynomials

$$[y_1, y_2, x_3], \tag{1}$$

where x_3 is a variable of even or odd degree, i. e., $x_3 = y_3$ or $x_3 = z_3$.

$$[y_1, z_2, y_3] \tag{2}$$

$$[y_1, z_2] \circ z_3 \tag{3}$$

$$[z_1 \circ z_2, z_3] \tag{4}$$

$$(z_1 \circ z_2)(z_3 \circ z_4) - (z_1 \circ z_3)(z_2 \circ z_4) \tag{5}$$

$$[x_1, y_2][y_3, x_4] + [x_1, y_3][y_2, x_4] \tag{6}$$

$$[y_1, z_2](z_3 \circ z_4) - [y_1, z_3](z_2 \circ z_4) \tag{7}$$

Lemma 4 *The graded identities from (1) to (7) hold for the algebra $E_{0^*} \otimes E_0$. In other words $I \subset T_2(E_{0^*} \otimes E_0)$.*

Proof. The proof consists of straightforward (and easy) computations, so we omit it. ■

Lemma 5 *The polynomial $[z_1 \circ z_2, y_3]$ lies in I .*

Proof. We have

$$[z_1 \circ z_2, y_3] = z_1 \circ [z_2, y_3] + z_2 \circ [z_1, y_3],$$

and the result follows from (3) since each summand in the right side of the above equality lies in I . ■

We denote by R the algebra $\mathbb{K}\langle Y \cup Z \rangle / I$, we denote by y_i (resp. z_i) the image of $y_i \in Y$ (resp. $z_i \in Z$) in the quotient R .

Lemma 6 *In the relatively free algebra R every polynomial $f(z_1, \dots, z_n)$ is a linear combination of the polynomials*

$$(z_{i_1} \circ z_{i_2}) \dots (z_{i_{2k-1}} \circ z_{i_{2k}}) z_{j_1} \dots z_{j_l}, \quad (8)$$

where $i_1 \leq i_2 \leq \dots \leq i_{2k}$ and $j_1 \leq j_2 \leq \dots \leq j_l$.

Proof. Let W be the subspace of R generated by the polynomials (8) and U be the subspace of R of the polynomials $f(z_1, \dots, z_n)$. Clearly $W \subset U$ we will prove the reverse inclusion by contradiction. We order the monomials $z_{j_1} \dots z_{j_l}$ by the lexicographic order, let $m = z_{j_1} \dots z_{j_l}$ be the least monomial that does not lie in W , in this case $j_1 \leq \dots \leq j_a$, and $j_a > j_{a+1}$ where $1 \leq a < l$, therefore

$$z_{j_1} \dots z_{j_l} = z_{j_1} \dots z_{j_{a-1}} (z_{j_a+1} \circ z_{j_a} - z_{j_a+1} z_{j_a}) \dots z_{j_l}.$$

It follows from (4) that

$$m = (z_{j_a+1} \circ z_{j_a})(z_{j_1} \dots z_{j_{a-1}} \dots z_{j_l}) - z_{j_1} \dots z_{j_{a-1}} z_{j_a+1} z_{j_a} \dots z_{j_l},$$

since $z_{j_1} \dots z_{j_{a-1}} \dots z_{j_l} < m$ and $z_{j_1} \dots z_{j_{a-1}} z_{j_a+1} z_{j_a} \dots z_{j_l} < m$ it follows that $z_{j_1} \dots z_{j_{a-1}} \dots z_{j_l}$ and $z_{j_1} \dots z_{j_{a-1}} z_{j_a+1} z_{j_a} \dots z_{j_l}$ lie in W , but using the identity (5) we conclude that $(z_{j_a+1} \circ z_{j_a})(z_{j_1} \dots z_{j_{a-1}} \dots z_{j_l})$ lies in W , a contradiction. ■

We consider the polynomials,

$$h_y(y_{i_1}, \dots, y_{i_{2m}}) = [y_{i_1}, y_{i_2}] \dots [y_{i_{2m-1}}, y_{i_{2m}}],$$

$$h_{y,z}(y_{i_{2m+1}}, \dots, y_{i_{2m+n}}, z_{j_1}, \dots, z_{j_n}) = [y_{i_{2m+1}}, z_{j_1}] \dots [y_{i_{2m+n}}, z_{j_n}],$$

and

$$h_z(z_{j_{n+1}}, \dots, z_{j_{n+2p}}) = (z_{j_{n+1}} \circ z_{j_{n+2}}) \dots (z_{j_{n+2p-1}} \circ z_{j_{n+2p}}).$$

Define

$$h(y_{i_1}, \dots, y_{i_{2m+n}}, z_{j_1}, \dots, z_{j_{n+2p}}) = h_y h_{y,z} h_z. \quad (9)$$

Let $K = \{k_1, \dots, k_q\}$ be a set of q natural numbers $k_1 < \dots < k_q$ we denote by g_K the product of the monomial $z_{k_1} \dots z_{k_q}$ by $h(y_{i_1}, \dots, y_{i_{2m+n}}, z_{j_1}, \dots, z_{j_{n+2p}})$.

Remark 7 Clearly using identities (5), (6) and (7) we may assume, multiplying by -1 if necessary, that in the polynomials g_K above we have $i_1 < i_2 < \dots < i_{2m+n}$, $j_1 \leq \dots \leq j_{n+2p}$, $k_1 < \dots < k_q$.

Lemma 8 In the relatively free algebra R every Y -proper polynomial is a linear combination of the polynomials

$$g_K = h(y_{i_1}, \dots, y_{i_{2m+n}}, z_{j_1}, \dots, z_{j_{n+2p}}) z_{k_1} \dots z_{k_q},$$

where $i_1 < i_2 < \dots < i_{2m+n}$, $j_1 \leq \dots \leq j_{n+2p}$, $k_1 < \dots < k_q$.

Proof. Let U be the subspace of R generated by the polynomials g_K . From the identity (1) we conclude that comutators $[y_a, y_b]$ lie in the center of R . It follows from (2) and (3) that the commutators $[y_a, z_b]$ commute with the even indeterminates and anticommute with the odd indeterminates. It is clear, using the identity (4) and Lemma 5, that the elements $z_a \circ z_b$ also lie in the center of R .

Therefore the product of two polynomials of the form g_K is a polynomial of the same type with the ordering of the indices following from identities (5), (7) and Lemma 8, thus U is a subalgebra of R . Moreover if f is a polynomial of the form g_K then $[f, x_i]$ lies in U for any indeterminate x_i , hence every commutator $[y_1, x_2, \dots, x_n]$ lies in U . ■

Recall that $A_0 = A_{(0,0)} \oplus A_{(1,0)}$ and $A_1 = A_{(0,1)} \oplus A_{(1,1)}$ and let $\varphi : \mathbb{K}\langle Y \cup Z \rangle \rightarrow A$ be a graded homomorphism. Given $y \in Y$ we denote by $\varphi(y)_0$ and $\varphi(y)_1$ the projections of $\varphi(y)$ in $A_{(0,0)}$ and $A_{(1,0)}$ respectively. Analogously one defines $\varphi(z)_0$ and $\varphi(z)_1$ for $z \in Z$.

Lemma 9 Let $\varphi : \mathbb{K}\langle Y \cup Z \rangle \rightarrow E_{0*} \otimes E_0$ be a graded homomorphism and let $h(y_{i_1}, \dots, y_{i_{2m+n}}, z_{j_1}, \dots, z_{j_{n+2p}})$ be the polynomial defined in (9). We have

$$\varphi(h) = \pm 2^{m+n} \left(\prod_{k=1}^{2m+n} \varphi(y_{i_k})_1 \right) \cdot \left(\prod_{l=1}^{n+2p} \varphi(z_{j_l})_1 \right).$$

Proof. Since $\varphi(y_{i_k})_0$ lies in the center of A and the elements $\varphi(y_{i_k})_1$ anticommute it follows that

$$\varphi([y_{i_1}, y_{i_2}] \dots [y_{i_{2m-1}}, y_{i_{2m}}]) = 2^m \prod_{k=1}^{2m} \varphi(y_{i_k})_1. \quad (10)$$

The elements $\varphi(y_{i_k})_1$ commute with $\varphi(z_{j_l})_0$ and anticommute with $\varphi(z_{j_l})_1$ hence

$$\varphi([y_{i_{2m+1}}, z_{j_1}] \dots [y_{i_{2m+n}}, z_{j_n}]) = \pm 2^n \left(\prod_{k=2m+1}^{2m+n} \varphi(y_{i_k})_1 \right) \cdot \left(\prod_{l=1}^{n+2p} \varphi(z_{j_l})_1 \right). \quad (11)$$

Moreover the elements $\varphi(z_{j_k})_0$ and $\varphi(z_{j_l})_1$ anticommute, therefore

$$\varphi((z_{j_{n+1}} \circ z_{j_{n+2}}) \dots (z_{j_{n+2p-1}} \circ z_{j_{n+2p}})) = \left(\prod_{k=n+1}^{n+2p} \varphi(z_{j_k})_1 \right). \quad (12)$$

Finally multiplying (10), (11) and (12) yields the result. ■

Lemma 10 *Let h be the polynomial in the previous lemma and $\varphi : \mathbb{K}\langle Y \cup Z \rangle \rightarrow A$ be a graded homomorphism such that $\varphi(z_{j_k})_1 = \sum_{l=1}^{n_k} \alpha_k b_l^k$, where b_k is in the canonical basis \mathbf{B} of $E_{0^*} \otimes E_0$ and $\alpha_k \in K$. If $\deg_{z_{j_k}} > n_k$ for some k then $\varphi(h) = 0$, moreover if $\deg_{z_{j_k}} = n_k$ for every k then*

$$\varphi(h) = \pm 2^{m+n} \left(\prod_{k=1}^{n+2p} \alpha_k n_k! \right) \cdot \left(\prod_{k=1}^{2m+n} \varphi(y_{i_k})_1 \right) \cdot \left(\prod_{k=1}^{n+2p} \left(\prod_{l=1}^{n_k} b_l^k \right) \right).$$

Proof. We have $\prod_{l=1}^{n+2p} \varphi(z_{j_l})_1 = \prod_{j_k} (\sum_{l=1}^{n_k} \alpha_k b_l^k)^{d_k}$, where d_k is the degree of z_{j_k} and the last product runs over all j_k such that z_{j_k} appears in h . Since the b_l^k commute and $(b_l^k)^2 = 0$ it follows that $(\sum_{l=1}^{n_k} \alpha_k b_l^k)^{n_k} = (n_k!) \prod_{l=1}^{n_k} \alpha_k b_l^k$ and $(\sum_{l=1}^{n_k} \alpha_k b_l^k)^{d_k} = 0$ if $d_k > n_k$, hence the result follows from the previous lemma. ■

If \mathbb{K} is a field of characteristic $p > 2$ let I_p denote the T_2 -ideal generated by the identities (1)-(7) together with the identities

$$[y_1, z_1] \dots [y_{2k-2}, z_1] (z_2 \circ z_1) z_1^{2n} \quad (13)$$

and the identities

$$[y_1, z_1] \dots [y_{2k-1}, z_1] z_1^{2n}, \quad (14)$$

where $2n + 2k - 1 = p$, $n = 0, 1, \dots, \frac{p-1}{2}$.

We denote by R_p the algebra $\mathbb{K}\langle Y \cup Z \rangle / I_p$.

Lemma 11 *If \mathbb{K} is a field of characteristic $p > 2$ then $I_p \subset T_2(E_{0^*} \otimes E_0)$.*

Proof. It follows from Lemma 9 that for any graded homomorphism $\varphi : \mathbb{K}\langle Y \cup Z \rangle \rightarrow A$ we have

$$\varphi([y_1, z_1] \dots [y_{2k-1}, z_1] z_1^{2n}) = \pm 2^{2k-1} \left(\prod_{i=1}^{2k-1} \varphi(y_{i_1})_1 \right) \cdot (\varphi(z_{j_l})_1)^p,$$

where $2n + 2k - 1 = p$. We have $\varphi(z_{j_l})_1 = \sum_{l=1}^n \alpha_l b_l$, where $b_l \in \beta$, in this case the elements b_l commute and $(b_l)^2 = 0$ for each l , hence

$$\left(\sum_{l=1}^n \alpha_l b_l \right)^p = \sum_{\substack{l_i \neq l_j \\ 1 \leq i < j \leq p}} b_{l_1} \dots b_{l_p} = \sum_{l_1 < \dots < l_p} (p!) b_{l_1} \dots b_{l_p} = 0.$$

Analogously $\varphi(z_1^{p+1}) = 0$, therefore $[y_1, z_1] \dots [y_k, z_1] z_1^{2n}$ and z_1^{p+1} are graded identities for $E_{0^*} \otimes E_0$ and now the result follows from Lemma 4. ■

Corollary 12 *In the algebra R_p every Y -proper polynomial is a linear combination of the polynomials*

$$g = h(y_{i_1}, \dots, y_{i_{2m+n}}, z_{j_1}, \dots, z_{j_{n+2p}}) z_{k_1} \dots z_{k_q},$$

where $i_1 < i_2 < \dots < i_{2m+n}$, $j_1 \leq \dots \leq j_{n+2p}$, $k_1 < \dots < k_q$ and $\deg_{z_l} h < p$, $1 \leq l \leq n + 2p$.

Proof. For each l if $\deg_{z_l} h \geq p$ then h is a consequence of (13) or (14), hence from Lemma 11 it follows that $h = 0$, now the result follows from Lemma 8. ■

Theorem 13 *If \mathbb{K} is a field of characteristic zero then the 2-graded identities of $E_0^* \otimes E_0$ follow from the identities (1) - (7). If \mathbb{K} is an infinite field of characteristic $p > 2$ then the 2-graded identities of $E_0^* \otimes E_0$ follow from the identities (1) - (7) together with the identities (13) and (14).*

Proof. Let \mathbb{K} be an infinite field of characteristic $p > 2$. Let $f(y_1, \dots, y_m, z_1, \dots, z_m)$ be a multihomogeneous Y -proper 2-graded identity for $E_0^* \otimes E_0$, then it follows from Corollary 12 that in R_p we have $f = \sum \alpha_K g_K$, where

$$g_K = h_K(y_1, \dots, y_m, z_1, \dots, z_n) z_{k_1} \dots z_{k_q},$$

and $K = \{k_1, \dots, k_q\}$. If $\alpha_K = 0$ for every subset K of $\{1, \dots, m\}$ we are done. If $\alpha_K \neq 0$ for some $K \subset \{1, \dots, m\}$, we denote by K_0 a minimal subset with $\alpha_{K_0} \neq 0$. Given $K \subset \{1, \dots, m\}$ let $\varphi_K : \mathbb{K}\langle Y \cup Z \rangle \rightarrow A$ be a 2-graded homomorphism such that $\varphi(z_j)_1 = \sum_{l=1}^{n_j} \alpha_j b_l^j$, where n_j is the degree of z_j in the polynomial h_K , $\varphi(z_j)_0 = 1 \otimes e_{a_j}$ and the elements $\varphi(y_i)$, $\varphi(z_j)_1$ and $\varphi(z_j)_0$ have disjoint supports.

Let K_1 be a subset of $\{1, 2, \dots, n\}$ such that $\alpha_{K_1} \neq 0$, since K_0 is minimal we have $K_0 \not\subset K_1$. Then there exists $k_a \in K_0$ such that $k_a \notin K_1$. Since g_{K_0} and g_{K_1} have the same multidegree we conclude that $\deg_{z_{k_a}}(h_{K_1}) > \deg_{z_{k_a}}(h_{K_0})$. Therefore $\varphi_{K_0}(z_{k_a})_1 = \sum_{l=1}^{n_{k_a}} \alpha_{k_a} b_l^{k_a}$ and $\deg_{z_{k_a}}(h_{K_1}) > n_{k_a} = \deg_{z_{k_a}}(h_{K_0})$, hence it follows from Lemma 10 that $\varphi_{K_0}(g_{K_1}) = 0$. Moreover $\deg_{z_j}(h_{K_0}) < p$ therefore it follows from Lemma 10 that $\varphi_{K_0}(g_{K_0}) \neq 0$. Hence we conclude that

$$0 = \varphi_{K_0}(f) = \alpha_{K_0} \varphi_{K_0}(g_{K_0}),$$

which is a contradiction. Hence $\alpha_K = 0$ for all $K \subset \{1, \dots, m\}$ and $f = 0$ in R_p , therefore $f \in I_p$.

Since f is an arbitrary Y -proper multihomogeneous polynomial we conclude that $T_2(E_0^* \otimes E_0) \subset I_p$ and the reverse inclusion follows from Lemma 11. The case where \mathbb{K} is a field of characteristic 0 is analogous. ■

4 The Graded Identities with an E_∞ Component

In this section consider the algebra A with one of the gradings $E_\infty \otimes E_k$, $E_\infty \otimes E_{j^*}$ or $E_\infty \otimes E_\infty$. Clearly the results in this section also hold for $E_k \otimes E_\infty$ and $E_{j^*} \otimes E_\infty$.

We denote by $\alpha(a) \in \mathbb{Z}_2$ the degree of a homogeneous element $a \in A$ and by $\alpha(f) \in \mathbb{Z}_2$ the degree of a homogeneous polynomial in $\mathbb{K}\langle X \rangle$. We recall that $g(a)$ denotes the degree of a homogeneous element in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of A . In this section prove that the graded identities of A are essentially the ordinary identities of $E \otimes E$.

Lemma 14 *Given $g_1, \dots, g_n \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and $h_1, \dots, h_n \in \mathbb{Z}_2$ there exists $b_1, \dots, b_n \in B$ with pairwise disjoint supports such that $g(b_i) = g_i$ and $\alpha(b_i) = h_i$.*

Proof. Clearly there exists $c_1, \dots, c_n \in \mathbf{B}$ with pairwise disjoint supports such that $g(c_i) = g_i$. Let $d_l = e_{a_{2l-1}} e_{a_{2l}} \otimes 1$, where a_1, \dots, a_{2n} is a sequence of pairwise different natural numbers such that the elements a_{2l} , $l = 1, 2, \dots, n$ are even and a_{2l-1} is even if and only if $h_l = \alpha(c_l)$. Moreover we may choose the elements a_i sufficiently large so that the elements $b_l = c_l d_l$ are different from 0 and the elements $\pm b_l \in \mathbf{B}$ have pairwise disjoint supports. In this case $g(b_l) = g(c_l) = g_l$ and $\alpha(b_l) = h_l$ and the lemma is proved. ■

Theorem 15 *If \mathbb{K} is an infinite field then $f(x_1, \dots, x_n)$ is a 2-graded identity for A if and only if $f(x_1, \dots, x_n)$ is an ordinary identity for A .*

Proof. Let $f \in T_2(A)$ be a multihomogeneous polynomial and a'_1, \dots, a'_n be arbitrary elements in $E \otimes E$. Let $a'_l = \sum_{k=1}^{n_l} \alpha_k^l c_k^l$, where $\alpha_k^l \in K$, $c_k^l \in \mathbf{B}$. It follows from Lemma 14 that there exists $b_k^l \in \mathbf{B}$ with pairwise disjoint supports such that $g(b_k^l) = g(c_k^l)$ and the elements $a_l = \sum_{k=1}^{n_l} \alpha_k^l b_k^l$ are homogeneous in A with $\alpha(a_l) = \alpha(x_l)$. Since f is a 2-graded identity for A we have $f(a_1, \dots, a_n) = 0$ and it follows from Proposition 3 that $f(a'_1, \dots, a'_n) = 0$. ■

5 The Graded Identities of $E_{k^*} \otimes E_{j^*}$

In this section we describe the graded identities of $E_{k^*} \otimes E_{j^*}$.

Lemma 16 *The polynomial $z_1 z_2 \dots z_{j+k+1}$ is a graded identity for $E_{k^*} \otimes E_{j^*}$.*

Proof. Since the polynomial is multilinear we need only to consider substitutions of the indeterminates z_i by elements b_i of the basis \mathbf{B} with degree 1 in the \mathbb{Z}_2 -grading of $E_{k^*} \otimes E_{j^*}$. In this case b_i must be divisible by an element of the set $\{e_1 \otimes 1, \dots, e_k \otimes 1, 1 \otimes e_1, \dots, 1 \otimes e_j\}$, hence it follows from the pigeon-hole principle that the elements b_1, \dots, b_{j+k+1} cannot have disjoint supports, therefore $b_1 b_2 \dots b_{j+k+1} = 0$. ■

Lemma 17 *Let $g_1, \dots, g_n \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and $h_1, \dots, h_n \in \mathbb{Z}_2$. If the number of indexes i in $\{1, 2, \dots, n\}$ with $h_i = 1$ is $\leq k + j$ then there exists $b_1, \dots, b_n \in \mathbf{B}$ with pairwise disjoint supports such that $g(b_i) = g_i$ and $\alpha(b_i) = h_i$.*

Proof. The proof is analogous to that of Lemma 14 and will be omitted. ■

Let I_m denote the T_2 -ideal generated by the identity $z_1 \dots z_{m+1}$ together with the polynomials $f(x_1, \dots, x_n) \in \mathbb{K}\langle X \rangle$ that are ordinary identities for $E \otimes E$.

Theorem 18 *If \mathbb{K} is an infinite field then $T_2(E_{k^*} \otimes E_{j^*}) = I_{j+k}$.*

Proof. It follows from Lemma 17 that $I_{k+l} \subset T_2(E_{k^*} \otimes E_{j^*})$. Let $f(x_1, \dots, x_n)$ be a multihomogeneous 2-graded identity for $E_{k^*} \otimes E_{j^*}$, modulo the identity $z_1 \dots z_{j+k+1}$ we may assume that the total degree of f in the odd indeterminates is $\leq k + l$. Hence if $a'_l = \sum_{k=1}^{n_l} \alpha_k^l c_k^l$, where $\alpha_k^l \in K$, $c_k^l \in \mathbf{B}$ and $n_l = \deg_{x_l}(f)$, it follows from Lemma 17 that there exists $b_k^l \in \mathbf{B}$ such that $g(b_k^l) = g(c_k^l)$ and the

elements $a_l = \sum_{k=1}^{n_l} \alpha_k^l b_k^l$ have disjoint support and are homogeneous in A with $\alpha(a_l) = \alpha(x_l)$. Since f is a 2-graded identity for A we have $f(a_1, \dots, a_n) = 0$ and it follows from Proposition 3 that $f(a'_1, \dots, a'_n) = 0$. ■

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